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VARIATIONAL OPTIMIZATION PROBLEMS FOR EQUATIONS OF HYPERBOLIC TYPE

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We consider the problems of optimizing control processes for systems described by second-order equations of hyperbolic type, posed in the form of the related two-dimensional Bolza problem of the calculus of variations. The necessary stationarity conditions are obtained. It is shown that Lagrange multipliers, which can have discontinuities inside the region of admissible variations, correspond to the optimal solutions.

Optimal problems for hyperbolic equations with conditions on the characteristics for functionals of the simplest form have been considered in [1, 2] by means of Pontriagin's maximum principle.

1. Statement of the problem. We consider a partial differential equation and relations given in a two-dimensional region Ω $(a \leqslant x \leqslant b,\ c \leqslant y \leqslant d)$ of the following form:

$$L(z) = a_{11}z_{xx} + a_{22}z_{yy} + a_1 z_x + a_2 z_y = f(x, y, z, u) \tag{1.1}$$

$$\psi_k(x, y, u) = 0 \qquad (k = 1, \ldots, r < m) \tag{1.2}$$

Here z_x, z_y, z_{xx}, z_{yy} are the first and second partial derivatives of the continuous function $z(x, y)$ being sought. By $u = (u_1(x, y), \ldots, u_m(x, y))$ we mean an m-dimensional vector of piecewise-continuous controls $u_k(x, y)$. The coefficients $a_1 = a_1(x, y)$, $a_2 = a_2(x, y)$, $a_{11} = a_{11}(x, y)$, $a_{22} = a_{22}(x, y)$ and the functions $f = f(x, y, z, u)$ and $\psi_k = \psi_k(x, y, u)$ are continuous and have continuous partial derivatives up to third order inclusive with respect to all the arguments for $x, y \in \Omega$. The initial and boundary conditions

$$z(a, y) = \varphi_1(y), \qquad z_x(a, y) = \varphi_2(y) \tag{1.3}$$

$$\varphi_c(x, z, z_y) = 0 \quad \text{for} \quad y = c \tag{1.4}$$
$$\varphi_a(x, z, z_y) = 0 \quad \text{for} \quad y = d$$

are taken as specified. In these equalities the functions $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_c(x, z, z_y)$ and $\varphi_d(x, z, z_y)$ are continuous and have continuous partial derivatives up to third order inclusive with respect to all the arguments.

We pose the following optimal problem: among the surfaces which inside region Ω satisfies Eqs. (1.1) and (1.2), satisfy relations (1.3) for $x = a$, and satisfy dependencies (1.4) for $y = c$ and $y = d$, find the one which minimizes the functional

$$J = \iint_{\Omega} f_0\, dx\, dy + \int_c^d \varphi_b\, dy + \chi(z^\circ(b, y)) \qquad (1.5)$$

Here $z^\circ(b, y) = (z(b, y_1^\circ), \ldots, z(b, y_p^\circ))$ is a p-dimensional vector, where y_k° are given numbers and $y_1^\circ = c$, $y_p^\circ = d$. The functions $f_0 = f_0(x, y, z, u)$ and $\chi(z^\circ(b, y))$ are continuous together with their derivatives up to third order inclusive with respect to the arguments. The function $\varphi_b = \varphi_b(y, z(b, y), z_x(b, y))$ is piecewise-continuous and, moreover, $\varphi_b(y, z, z_x) = \varphi_{b\gamma}(y, z, z_x)$ for $y \in (y_\gamma, y_{\gamma+1})$ and $\varphi_{b\gamma}(y, z, z_x)$ is continuous together with its derivatives up to third order inclusive. The discontinuities of the function $\varphi_b(y, z, z_x)$ at $y = y_\gamma$ are taken as specified.

2. Necessary condition for the stationarity of J. The Euler equation. For the stated problem we can prove lemmas on the imbedding of the surface E minimizing functional (1.5) into a one-parameter or multiparameter family of comparison surfaces. The necessary condition for the stationarity of functional J can be proved with the aid of these lemmas. Here it is used in the same form as in [3, 4] for the related one-dimensional Bolza problem and in [5] for the multidimensional problem.

For the functional J to take a minimal value on a surface E it is necessary to fulfill on it the equality

$$\Delta I = 0 \qquad (2.1)$$

in which

$$I = I_0 + I_1 + I_2 = \chi(z^\circ(b, y)) + \int_a^b L_1'\, dx + \int_c^d L_1''\, dy + \iint_{\Omega} L_2\, dx\, dy \qquad (2.2)$$

$$L_1'(z_y, z, \eta_c, \eta_d, x) = \eta_c \varphi_c(x, z, z_y) + \eta_d \varphi_d(x, z, z_y) \qquad (2.3)$$

$$L_1''(z_x, z, \eta_1, \eta_2, y) = \eta_1 [z(a, y) - \varphi_1(y)] + \eta_2 [z_x(a, y) - \varphi_2(y)]$$

$$L_2(z_{xx}, z_{yy}, z_x, z_y, u, \lambda, \mu, x, y) = f_0 + \lambda L(z) - \lambda f + \sum_{k=1}^r \mu_k \psi_k$$

where

$\lambda = \lambda(x, y)$, $\mu_k = \mu_k(x, y)$, $\eta_c = \eta_c(x)$, $\eta_d = \eta_d(x)$, $\eta_1 = \eta_1(y)$, $\eta_2 = \eta_2(y)$ are undetermined Lagrange multipliers, ΔI is the total variation of functional I.

To compute the variation ΔI we take it that the whole region Ω consists of n elementary regions ω_i $(i = 1, \ldots, n)$; in each of them the functions $z(x, y)$ and $\lambda(x, y)$ are continuous and have continuous derivatives, and the functions $\mu_k(x, y)$ and $u_1(x, y), \ldots, u_m(x, y)$ are continuous. The elementary region ω_i has a piecewise-smooth boundary S_i. The smooth segments S_{ij} $(j = 1, \ldots, \tau_i)$ of this boundary can be lines of the following types: (1) a part of the boundary of region Ω, (2) a line of discontinuity of the control parameters, not coincident with the characteristic of Eq. (1.1), (3) a line of discontinuity of the control parameters, coinciding with the characteristic of Eq. (1.1), (4) a line of discontinuity of the multipliers $\lambda(x, y)$, $\mu_k(x, y)$, coinciding with the characteristic of (1.1). The number of noncharacteristic boundary lines, interior relative to region Ω, is denoted by q_1, while the number of characteristic boundary lines, by q_2. We introduce the notation $q = q_1 + q_2$. We assume the boundary S_i has τ_i points M_{ij} where the smoothness is violated. At each of them any finite number of noncharacteristic boundary lines can intersect with one or two characteristic boundary lines. Let m_a, m_b, m_c and m_d be the number of elementary regions ω_i abutting on the parts $x = a$, $x = b$, $y = c$ and $y = d$ of the boundary of region Ω.

Let us consider the individual terms on the left-hand side of equality (2.1). We begin with the variation ΔI_2. Setting it up we have

$$\Delta I_2 = \iint_{\Omega} \Big[\frac{\partial L_2}{\partial z} \delta z + \lambda L(\delta z) + \sum_{k=1}^{m} \frac{\partial L_2}{\partial u_k} \delta u_k \Big] dx\, dy + \sum_{i=1}^{n} \oint_{S_i} L_2 \delta N\, ds \quad (2.4)$$

Here δz, δu_k and δN are the variations of the functions z, u_k and of the boundary contour S_i in the normal direction. After applying the Green-Riemann formula to the integral containing $L\ (\delta z)$ and using formula (2.3), we obtain

$$\begin{aligned} \Delta I_2 = \sum_{i=1}^{n} \iint_{\omega_i} \Big\{ \Big[&(a_{11}\lambda_i)_{xx} + (a_{22}\lambda_i)_{yy} - (a_1\lambda_i)_x - (a_2\lambda_i)_y - \frac{\partial f}{\partial z}\lambda_i + \frac{\partial f_0}{\partial z} \Big] \delta z_i + \\ &+ \sum_{k=1}^{m} \Big[\sum_{\alpha=1}^{r} \mu_{\alpha i} \frac{\partial \psi_\alpha}{\partial u_{ki}} + \frac{\partial f_0}{\partial u_k} - \lambda_i \frac{\partial f}{\partial u_k} \Big] \delta u_{ki} \Big\} dx\, dy + \qquad (2.5) \\ &+ \sum_{i=1}^{n} \int_{S_i} \{ [a_1\lambda_i \delta z_i + a_{11}\lambda_i \delta z_{ix} - (a_{11}\lambda_i)_x \delta z_i]\, n_{1i} + \\ &+ [a_2\lambda_i \delta z_i - a_{22}\lambda_i \delta z_{iy} + (a_{22}\lambda_i)_y \delta z_{iy}]\, n_{2i} + L_2 \delta N \}\, ds \end{aligned}$$

$$n_1 = \frac{dx}{dN} = \frac{dy}{ds}, \qquad n_2 = \frac{dy}{dN} = -\frac{dx}{ds}$$

Here n_1, n_2 are the direction cosines of the normal to contour S; N and s are coordinates counted off along the normal and along the tangent to the contour (the tangent is directed toward the side of the positive circuit of the contour, the normal is taken outward); δN is the variation of the contour S_i along the normal direction; the index i denotes the membership of the corresponding functions in the elementary region ω_i.

Computing the variations ΔI_1 and ΔI_0, we find

$$\Delta I_1 = \int_c^d (\eta_1 \delta z + \eta_2 \delta z_x)\, dy + \int_a^b \eta_c \Big(\frac{\partial \varphi_c}{\partial z} \delta z + \frac{\partial \varphi_c}{\partial z_y} \delta z_y \Big) dx + \qquad (2.6)$$

$$+ \int_a^b \eta_d \Big(\frac{\partial \varphi_d}{\partial z} \delta z + \frac{\partial \varphi_d}{\partial z_y} \delta z_y \Big) dx + \int_c^d \Big(\frac{\partial \varphi_b}{\partial z} \delta z + \frac{\partial \varphi_b}{\partial z_y} \delta z_y \Big) dy + \sum_{k=1}^{m_b} (\varphi_{bk}^{-} - \varphi_{bk}^{+})\, \Delta y_k$$

$$\Delta I_0 = \sum_{\gamma=1}^{p} \frac{\partial \chi}{\partial z_\gamma} \delta z\, (b, y_\gamma^\circ) \quad (2.7)$$

The minus and plus superscripts denote the left and right limits of the function φ_b. Substituting ΔI_0, ΔI_1 and ΔI_2 into equality (2.1), we obtain an expression containing terms depending on double integrals over the elementary regions ω_i, on integrals along the boundaries S_i of these regions, and on integrals along parts of the boundary of region Ω, and terms independent of integrals. The usual arguments of the calculus of variations allow us to establish that to fulfill the stationarity condition (2.1) we need to equate each of these groups of terms to zero. Equating to zero the terms containing the multiple integrals and applying the fundamental lemma of the calculus of variations, we obtain the Euler equation

$$M(\lambda) = (a_{11}\lambda)_{xx} + (a_{22}\lambda)_{yy} - (a_1\lambda)_x - (a_2\lambda)_y = (\partial f / \partial z)\lambda - \partial f_0 / \partial z \qquad (2.8)$$

determining the multiplier $\lambda(x, y)$, and the relations

$$\sum_{\alpha=1}^{r} \mu_\alpha \frac{\partial \psi_\alpha}{\partial u_k} + \frac{\partial f_0}{\partial u_k} - \lambda \frac{\partial f}{\partial u_k} = 0 \qquad (k = 1, \ldots, m) \qquad (2.9)$$

which must be fulfilled in each of the elementary regions ω_i, i.e. at each point of region Ω at which $\lambda(x, y)$ and $\mu_\alpha(x, y)$ are continuous. Here the indices i have been omitted in Eqs. (2.8) and (2.9).

3. The Erdmann-Weierstrass conditions. To obtain the Erdmann-Weierstrass conditions on the boundary lines S_i of the elementary regions ω_i and the boundary conditions at the boundary points of region Ω we analyze the remaining terms in the variation ΔI. At first we pass in ΔI to the coordinates s and N counted off along the tangent and along the normal to the boundary contour. Then for the derivatives of some function $F(x, y)$ we have the formulas $F_x = F_N n_1 - F_s n_2$, $F_y = F_N n_2 + F_s n_1$. We apply them for the computation of the derivatives δz_x, δz_y, $(a_{11}\lambda)_x$, $(a_{22}\lambda)_y$ occurring in relation (2.5) and we integrate by parts the terms containing the derivative δz_s. After carrying out these operations the expression for the variations takes the form

$$\begin{aligned}\Delta I = \sum_{i=1}^{n} \sum_{j=1}^{\tau_i} \int_{S_{ij}} \{A_{1ij}\lambda_{ij}\delta z_{ijN} + [(a_1 n_{1ij} + a_2 n_{2ij})\lambda_{ij} - (A_{1ij}\lambda_{ij})_N - 2(A_{2ij}\lambda_{ij})_s + \\ + (A_{3ij} - A_{1ij})\rho_{ij}^{-1}]\delta z_{ij} + [f_0 + \lambda_{ij}L(z_{ij}) - \lambda_{ij}f]\delta N_{ij}\} ds + \\ + \sum_{i=1}^{n} \sum_{j=1}^{\tau_i} [A_2\lambda\delta z]_{M_{ij}}^{M_{i,j+1}} + \int_a^b \left[\eta_c \frac{\partial \varphi_c}{\partial z} \delta z(x, c) + \eta_c \frac{\partial \varphi_c}{\partial z_y} \delta z_y(x, c)\right] dx + \qquad (3.1) \\ + \int_a^b \left[\eta_d \frac{\partial \varphi_d}{\partial z} \delta z(x, d) + \eta_d \frac{\partial \varphi_d}{\partial z_y} \delta z_y(x, d)\right] dx + \int_c^d [\eta_1 \delta z(a, y) + \\ + \eta_2 \delta z_x(a, y)] dy + \int_c^d \left[\frac{\partial \varphi_b}{\partial z} \delta z(b, y) + \frac{\partial \varphi_b}{\partial z_x} \delta z_x(b, y)\right] dy + \\ + \sum_{k=1}^{m_b} (\varphi_{bk}^- - \varphi_{bk}^+)\Delta y_k + \sum_{\gamma=1}^{p} \frac{\partial \chi}{\partial z_\gamma} \delta z(b, y_\gamma^\circ) = 0\end{aligned}$$

Here we have introduced the notation A_1, A_2 and A_3 defined by formulas (A.3) (see the Appendix) in which $a_{12} = 0$ and by ρ_i we have denoted the radius of curvature of contour S_i. We transform the variations δz_{ij}, δz_{ijN}, $\delta z|_{M_{ij}}$ occurring in relation (3.1), by the formulas

$$\delta z_{ij} = \Delta z_{ij} - z_{ijN}\delta N_{ij}, \; \delta z_{ijN} = \Delta z_{ijN} - z_{ijNN}\delta N \qquad (3.2)$$

$$\delta z|_{M_{ij}} = \Delta z|_{M_{ij}} - z_{ijx}\Delta x_{ij} - z_{ijy}\Delta y_{ij} \qquad (3.3)$$

where Δz_{ij} are the variations of function z on the line S_{ij}, $\Delta z|_{M_{ij}}$ is the variation of this function at the point M_{ij}, and Δx_{ij} and Δy_{ij} are the variations of the coordinates of point M_{ij} Making use of equalities (3.2) and (3.3), we reduce expression (3.1) to the form

$$\Delta I = \sum_{i=1}^{n} \sum_{j=1}^{\tau_i} \int_{S_{ij}} \{A_1\lambda\Delta z_N + [(a_1 n_1 + a_2 n_2)\lambda - (A_1\lambda)_N - 2(A_2\lambda)_s +$$

$$+ (A_3 - A_1)\rho^{-1}\lambda]\,\Delta z + [f_0 - \lambda f + 2A_2\lambda(z_{sN} - \rho^{-1}z_s) + \qquad (3.4)$$
$$+ A_3\lambda(z_{ss} + \rho^{-1}z_N) - (a_1n_2 - a_2n_1)\lambda z_s + (A_1\lambda)_N z_N +$$
$$+ 2(A_2\lambda)_s z_N + (A_1 - A_3)\rho^{-1}\lambda z_N]\,\delta N\}\,ds +$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{\tau_i}[A_2\lambda\Delta z - A_2\lambda(z_x\Delta x + z_y\Delta y)]_{M_{ij}}^{M_{ij+1}} +$$
$$+ \sum_{k=1}^{m_c}\int_{x_k}^{x_{k+1}}\eta_c\left[\frac{\partial\varphi_c}{\partial z}\Delta z(x, c) + \frac{\partial\varphi_c}{\partial z_y}\Delta z_y(x, c)\right]dx +$$
$$+ \sum_{k=1}^{m_d}\int_{x_k}^{x_{k+1}}\eta_d\left[\frac{\partial\varphi_d}{\partial z}\Delta z(x, d) + \frac{\partial\varphi_d}{\partial z_y}\Delta z_y(x, d)\right]dx +$$
$$+ \sum_{k=1}^{m_a}\int_{y_k}^{y_{k+1}}[\eta_1\Delta z(a, y) + \eta_2\Delta z_x(a, y)]\,dy + \sum_{k=1}^{m_b}\int_{y_k}^{y_{k+1}}\left[\frac{\partial\varphi_b}{\partial z}\Delta z(b, y) + \right.$$
$$\left. + \frac{\partial\varphi_b}{\partial z_x}\Delta z_x(b, y)\right]dy + \sum_{k=1}^{m_b}(\varphi_{bk}^{-} - \varphi_{bk}^{+})\,\Delta y_k + \sum_{\gamma=1}^{p}\frac{\partial\chi}{\partial z_\gamma}\Delta z(b, y_\gamma^\circ) = 0$$

Here and subsequently we have dropped the indices i and j and have used the equality

$$-(a_1n_1 + a_2n_2)\lambda z_N = -\lambda a_1 z_x - \lambda a_2 z_y - \lambda(a_1n_2 - a_2n_1)z_s \qquad (3.5)$$
$$-A_1\lambda z_{NN} = -\lambda a_{11}z_{xx} - \lambda a_{22}z_{yy} + 2A_2\lambda(z_{sN} - \rho^{-1}z_s) +$$
$$+ A_3\lambda(z_{ss} - \rho^{-1}z_N)$$

and, on the basis of condition (2.1), have required that the variation be equated to zero.

We go on to establish the Erdmann-Weierstrass conditions on the different parts of the boundary lines S_{ij} of the elementary regions ω_i. Let us consider a line of the second type and assume that it demarcates regions ω_i and ω_k. Functions relating to region ω_i are marked by a superscript minus, while functions relating to region ω_k, by a superscript plus. Then, on passing through the line S_{ij},

$$\Delta z^- = \Delta z^+ = \Delta z, \qquad \Delta z_N^- = \Delta z_N^+ = \Delta z_N \qquad (3.6)$$
$$\delta N^- = \delta N^+ = \delta N$$

(these variations are independent). In (3.4) we pick out the terms containing Δz_N and equate them to zero: we obtain $\lambda^- A_1^- - \lambda^+ A_1^+ = 0$. On a line of the second type $A_1^- = A_1^+ \neq 0$. Therefore,

$$\lambda^- = \lambda^+ \text{ on } S_{ij} \qquad (3.7)$$

Having picked out in (3.4) the terms depending on Δz, we find

$$-2A_2\lambda_s^- - A_1\lambda_N^- + B_1\lambda^- = -2A_2\lambda_s^+ - A_1\lambda_N^+ + B_1\lambda^+ \qquad (3.8)$$
$$B_1 = a_1n_1 + a_2n_2 - A_{1N} - 2A_{2s} - \rho^{-1}(A_1 - A_3)$$

Equality (3.7) and the relation $\lambda_s^- = \lambda_s^+$ are valid on the line S_{ij}. Consequently, the equality

$$\lambda_N^- = \lambda_N^+ \text{ on } S_{ij} \qquad (3.9)$$

is fulfilled. Finally, if in (3.4) we pick out the terms containing δN and take into account the conditions obtained above, we obtain the relation

$$f_0^- - \lambda^- f^- = f_0^+ - \lambda^+ f^+ \quad \text{on } S_{ij} \tag{3.10}$$

It should be noted that the terms, occurring in variation (3.4), depending on Δz, Δx and Δy, computed at the points M_{ij} for lines S_{ij} of the second type, cancel each other out.

Let us study a segment S_{ij} of the third type. Since S_{ij} is a segment of a characteristic, the equality $A_1 = 0$ is fulfilled on it and the terms in (3.5) containing Δz_N vanish. In this connection the nonequality $\lambda^- \neq \lambda^+$ can hold on S_{ij}. Equating the coefficients of variation Δz to zero, we obtain the equation

$$-2A_2[\lambda^- - \lambda^+]_s + B_1[\lambda^- - \lambda^+] = 0 \quad \text{on } S_{ij} \tag{3.11}$$

determining the change $\lambda^- - \lambda^+$ in the discontinuity of the multiplier λ along the characteristic. Consequently, this discontinuity can be found if we are given the conditions on the boundary of the region Ω, which we shall obtain subsequently. If the line to be studied is fixed, then $\delta N = 0$. In the case of a moving line the conditions have a rather complicated form and are studied below.

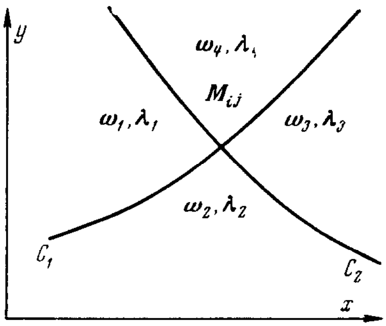

Fig. 1

Let us consider the point M_{ij} of intersection of two characteristics C_1 and C_2 (Fig. 1). Any number of noncharacteristic lines can intersect at this point; they do not introduce additional discontinuities in multiplier λ, therefore, we can use the notation shown in Fig. 1. Picking out from (3.4) the terms relating to the point M_{ij}, after cancelling the nonzero multiplier $A_{2C_1}^{\bullet} - A_{2C_2}$, we obtain the expression

$$[\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4]_{M_{ij}} = 0 \tag{3.12}$$

proving that the magnitude of the discontinuity of multiplier λ on a characteristic does not change on passing through another characteristic.

4. Boundary conditions. Let us now consider lines S_{ij} of the first type, which are parts of the outer boundary of region Ω. We start with the boundary $x = a$, $c \leqslant y \leqslant d$. Equating the coefficients of the variations $\Delta z_x\ (a, y)$ and $\Delta z\ (a, y)$ to zero, we obtain two conditions apiece,

$$A_{1a}\lambda_k(a, y) = \eta_{2k} \qquad (k = 1, \ldots, m_a) \tag{4.1}$$
$$-A_{1a}\lambda_{kx}(a, y) + (a_1 - A_{1ax})\lambda_k(a, y) = \eta_{1k}$$

Analysis of the terms containing the variations $\Delta z_y\ (x, c)$ and $\Delta z\ (x, c)$ for the boundary segment $y = c$, $a \leqslant x \leqslant b$, leads to the following results :

$$A_{1c}\lambda_k(x, c) = \eta_{ck}\partial\varphi_c / \partial z_y \qquad (k = 1, \ldots, m_c)$$
$$A_{1c}\lambda_{ky}(x, c) - (a_2 - A_{1cy})\lambda_k(x, c) = -\eta_{ck}\partial\varphi_c / \partial z \tag{4.2}$$

If the points of intersection of the characteristic lines C_1 and C_2 shown in Fig. 2 occur on the boundary $y = c$, then we obtain the following relation for the discontinuities of multiplier λ, which must be fulfilled at $x = x_k$, $y = c$:

$$\lambda_\alpha - \lambda_k = \begin{cases} -\lambda_\alpha + \lambda_{k-1}, & \varphi_{cz_y}|_{x=x_k} \neq 0 \\ \lambda_\alpha - \lambda_{k-1}, & \varphi_{cz_y}|_{x=x_k} = 0 \end{cases} \tag{4.3}$$

Here we have used formula (A.10) from the Appendix. Analogous conditions are found for the boundary segment $y = d,\ a \leqslant x \leqslant b$. At all its points we have

$$A_{1d}\lambda_k(x, d) = \eta_{dk}\,\partial\varphi_d / \partial z_y \qquad (k = 1, \ldots, m_d) \tag{4.4}$$
$$A_{1d}\lambda_{ky}(x, d) - (a_2 - A_{1dy})\,\lambda_k(x, d) = -\eta_{dk}\,\partial\varphi_d / \partial z$$

At the points $x = x_k$ of intersection of the characteristics we obtain

$$\lambda_x - \lambda_k = \begin{cases} -\lambda_x + \lambda_{k-1}, & \varphi_{dz_y}|_{x=x_k} \neq 0 \\ \lambda_x - \lambda_{k-1}, & \varphi_{dz_y}|_{x=x_k} = 0 \end{cases} \tag{4.5}$$

The following conditions are obtained for the boundary segment $x = b,\ c \leqslant y \leqslant d$:

$$A_{1b}\lambda_k(b, y) = -\partial\varphi_{bk} / \partial z_x \qquad (k = 1, \ldots, m_b) \tag{4.6}$$
$$-A_{1b}\lambda_{kx}(b, y) + (a_1 - A_{1bx})\,\lambda_k(b, y) = -\partial\varphi_{bk} / \partial z$$

The above-mentioned given points

$$y = y^\circ_\gamma\ (\gamma = 1, \ldots, p), \qquad y_1^\circ = c, \quad y_p^\circ = d$$

occur on this segment and a certain number of moving points may appear. We number this as well as the other points from one to m_b, where $y_1 = c$, and $y_{m_b} = d$ (Fig. 3).

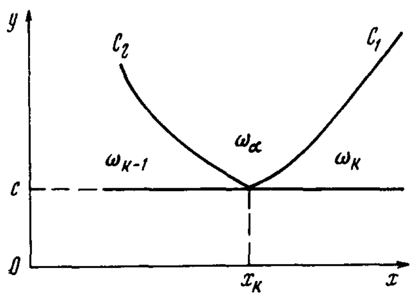

Fig. 2

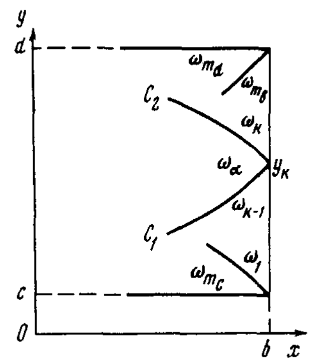

Fig. 3

Analysis of the terms in expression (3.4), corresponding to $y = y_k,\ x = b\ (k = 2, \ldots, m_b - 1)$, leads to the conditions

$$\lambda(b, y_k) = \frac{1}{2}\lambda(b, y_k - 0) + \frac{1}{2}\lambda(b, y_k + 0) -$$
$$-\frac{1}{2\sqrt{-a_{11}a_{22}}}\frac{\partial\chi}{\partial z(b, y_k)} \qquad (k = 2, \ldots, m_b - 1) \tag{4.7}$$

For the point $y_1 = y_1^\circ = c,\ x = b$ these conditions are replaced by the following:

$$\lambda(b - 0, c) = \lambda(b, c + 0) - \frac{1}{\sqrt{-a_{11}a_{22}}}\frac{\partial\chi}{\partial z(b, y_1^\circ)}, \qquad \frac{\partial\varphi_c}{\partial z_y} \neq 0 \tag{4.8}$$

Analogously, at the point $y_{m_b} = y_p^\circ = d,\ x = b$

$$\lambda(b - 0, d) = \lambda(b, d - 0) - \frac{1}{\sqrt{-a_{11}a_{22}}}\frac{\partial\chi}{\partial z(b, y_p^\circ)}, \qquad \frac{\partial\varphi_d}{\partial z_y} \neq 0 \tag{4.9}$$

If the derivatives $\partial\varphi_c/\partial z_y = 0$ or $\partial\varphi_d/\partial z_y = 0$, then at the corner points the muliplier λ can have, respecrively, the discontinuity $\lambda(b - 0,\ c) \neq \lambda(b,\ c + 0)$ or

$\lambda(b-0,\ d)\neq\lambda(b,\ d-0)$.

Let us find the necessary condition in the case of moving characteristics. From (3.4) we write out the remaining terms and we take the terms containing Δx and Δy under the integral, for which we use formulas (A.17) and (A.20). After the reduction of similar terms we obtain

$$\sum_{i=1}^{n_0}\sum_{j=1}^{\tau_{0i}}\int\limits_{S_{ij}}\left[f_0-\frac{1}{2}(\lambda^++\lambda^-)f\right]\delta N\,ds+$$
$$+\sum_{k=1}^{m_b}[a_{11}(b,\ y_k)\lambda(b,\ y_k)(z_x^- - z_k^+)+\varphi_b^- - \varphi_b^+]\,\Delta y_k=0 \qquad (4.10)$$
$$z_x^- = z_x(b,\ y_x-0)',\qquad z_x^+ = z_x(b,\ y_k+0)$$

Here n_0 is the number of elementary regions formed by the characteristics, τ_{0i} is the number of smooth characteristic lines bounding the ω_i, $\lambda^-=\lambda(x,y-0)$, $\lambda^+=\lambda\cdot(x,y+0)$, where x and y lie on the characteristic S_{ij}.

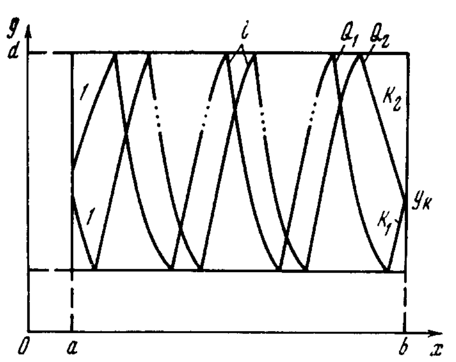

Fig. 4

Assume that there exists one moving point $y_k\neq y_\gamma^0$ ($\gamma=1,\ \ldots,\ p$). Then region Ω is divided by lines Q_1 and Q_2 made up, respectively, from k_1 and k_2 segments of the characteristics (Fig. 4). By x_i' we denote the intersection of the ith segment of the characteristic line Q_1 with the boundary $y=d$ if k_1-i is odd, or with the boundary $y=c$ if k_1-i is even ($i=2,\ \ldots,\ k_1$), and by x_j'', the intersection of the jth segment of characteristic line Q_2 with the boundary $y=c$ if k_2-j is odd, or with the boundary $y=d$ if k_2-j is even ($j=2,\ \ldots,\ k_2$).

In expression (4.10), δN depends, on each part of S_{ij}, on the variation of the constants D_1 and D_2 (see (A.16)). The lines Q_1 and Q_2 are continuous and issue from the point $x=b,\ y=y_k$, therefore, in the end each of their segments is determined by this point. The variation of the $(k_\alpha-i+1)$th segment of the characteristic line Q_α ($\alpha=1,\ 2$) has the form

$$\delta D_{\alpha k_\alpha-i+1}=\Theta_{\alpha,i}\Delta y_k\qquad (i=1,\ldots,\ k_\alpha) \qquad (4.11)$$

Here

$$\Theta_{1,i}=\begin{cases} F_{1y}(b,\ y_k) & (i=k_1)\\[2mm] \dfrac{F_{1x}(x_{i+1}',\ d)F_{2x}(x_{i+2}',\ c)\ldots F_{2x}(x_{k_1}',\ c)}{F_{2x}(x_{i+1}',\ d)F_{1x}(x_{i+2}',\ c)\ldots F_{1x}(x_{k_1}',\ c)}F_{1y}(b,\ y_k) & (k_1-i \text{ is even})\\[4mm] \dfrac{F_{2x}(x_{i+1}',\ c)F_{1x}(x_{i+2}',\ d)\ldots F_{2x}(x_{k_1}',\ c)}{F_{1x}(x_{i+1}',\ c)F_{2x}(x_{i+2}',\ d)\ldots F_{1x}(x_{k_1}',\ c)}F_{1y}(b,\ y_k) & (k_1-i \text{ is odd})\end{cases} \qquad (4.12)$$

$$\Theta_{2,i}=\begin{cases} F_{2y}(b,\ y_k) & (i=k_2)\\[2mm] \dfrac{F_{2y}(x_{i+1}'',\ c)F_{1x}(x_{i+2}'',\ d)\ldots F_{1x}(x_{k_2}'',\ d)}{F_{1x}(x_{i+1}'',\ c)F_{2x}(x_{i+2}'',\ d)\ldots F_{2x}(x_{k_2}'',\ d)}F_{2y}(b,\ y_k) & (k_2-i \text{ is even})\\[4mm] \dfrac{F_{1x}(x_{i+1}'',\ d)F_{2x}(x_{i+2}'',\ c)\ldots F_{1x}(x_{k_2}'',\ d)}{F_{2x}(x_{i+1}'',\ d)F_{1x}(x_{i+2}'',\ c)\ldots F_{2x}(x_{k_2}'',\ d)}F_{2y}(b,\ y_k) & (k_2-i \text{ is odd})\end{cases} \qquad (4.13)$$

Equating the coefficients of Δy_k to zero, we obtain the last necessary condition

$$\sum_{i=1}^{k_1} \Theta_{1,i} \int_{x_i'}^{x_{i+1}'} \left[f_0^+ - f_0^- - \frac{1}{2}(\lambda^- + \lambda^+)(f^+ - f^-) \right] \frac{dx}{\theta_1} +$$

$$+ \sum_{i=1}^{k_2} \Theta_{2,i} \int_{x_i''}^{x_{i+1}''} \left[f_0^+ - f_0^- - \frac{1}{2}(\lambda^- + \lambda^+)(f^+ - f^-) \right] \frac{dx}{\theta_2} = \qquad (4.14)$$

$$= \varphi_b^+ - \varphi_b^- + a_{11}(b, y_k)\lambda(b, y_k)(z_x^+ - z_x^-) = 0$$

Here $x_1' = x_1'' = a, \quad x_{k_1+1}' = x_{k_2+1}'' = b$

$$\theta_1 = \begin{cases} |F_{1y}| & (k_1 - i \text{ is even}) \\ |F_{2y}| & (k_1 - i \text{ is odd}) \end{cases}$$

$$\theta_2 = \begin{cases} |F_{1y}| & (k_2 - i \text{ is odd}) \\ |F_{2y}| & (k_2 - i \text{ is even}) \end{cases}$$

Appendix. We consider the following hyperbolic differential equation [6, 7]:

$$a_{11}z_{xx} + a_{12}z_{xy} + a_{22}z_{yy} + a_1 z_x + a_2 z_y = f(x, y, z, u) \qquad (A.1)$$

Here we have used the notation of Sect. 1. We pass to the new variables s and N counted off along the tangent and the normal to curve C. We obtain

$$A_1 z_{NN} + 2A_2 z_{sN} + A_3 z_{ss} + Bz_N + (-a_1 n_2 + a_2 n_1 - 2\rho^{-1}A_2) z_s = f \qquad (A.2)$$

$$A_1 = a_{11}n_1^2 + a_{12}n_1 n_2 + a_{22}n_2^2$$

$$A_2 = -a_{11}n_1 n_2 + {}^1/_2 a_{12}(n_1^2 - n_2^2) + a_{22}n_1 n_2 \qquad (A.3)$$

$$A_3 = a_{11}n_2^2 - a_{12}n_1 n_2 + a_{22}n_1^2$$

$$B = a_1 n_1 + a_2 n_2 + \rho^{-1}A_3$$

Here ρ is the radius of curvature of C ; n_1 and n_2 are the direction cosines of the normal. By z^- and z^+ we denote the value of function z to the left and to the right of curve C as we move along it in the direction of increasing s. Let z together with its derivative z_N be continuous on passing through C. Then, the derivatives z_s, z_{sN}, z_{ss} are continuous. For z_{NN} we have

$$A_1 [z_{NN}] = [f], \ [z_{NN}] = z_{NN}^- - z_{NN}^+, \ [f] = f^- - f^+ \qquad (A.4)$$

Consequently, $[z_{NN}] \neq 0$ if $[f] \neq 0$ and line C is not a solution of the equation

$$A_1 = a_{11}\left(\frac{dy}{ds}\right)^2 - a_{12}\frac{dx}{ds}\frac{dy}{ds} + a_{22}\left(\frac{dx}{ds}\right)^2 = 0 \qquad (A.5)$$

i.e. is not a characteristic of Eq. (A.1). For a continuous right-hand side of Eq. (A.1) the derivative z_{NN} can have a finite or an infinite discontinuity only on a characteristic line.

Having differentiated Eq. (A.2) with respect to N under the condition that C is a characteristic, we obtain the equation

$$2A_2 [z_{NN}]_s + B [z_{NN}] = [f_N] \qquad (A.6)$$

showing that breaks in the continuity of the function z_{NN} can arise from the boundary conditions and from the discontinuity of the function f_N. If z is continuous but z_N is discontinuous on passing through the line C, this line must be a characteristic, and the magnitude of the discontinuity $[z_N]$ satisfies the differential equation

$$2A_2 [z_N]_s + B [z_N] = [f] \tag{A.7}$$

Thus the source of the discontinuity of function z_N may be both the boundary conditions as well as the discontinuity of the right-hand side. If the function z is discontinuous, then for the magnitude of the discontinuity $[z]$ we obtain the equation

$$2A_2 [z]_s + (B - A_{1N}) [z] = 0 \tag{A.8}$$

showing that the discontinuities of function z can arise only as a result of the boundary conditions. A formula analogous to (A.8) (without the term containing A_{1N}) occurs in [7].

The coefficient A_2 of the derivative in Esq. (A.6)–(A.8) for the hyperbolic equation (A.1) for which $a^2_{12} - 4a_{11}a_{22} > 0$, is nonzero on characteristics. The equality $A_2 = 0$ defines two families of curves which may be taken as coordinate lines. In this case, instead of Eq. (A.1) we have Eq. (1.1) which we study subsequently. For it the equations for the characteristic have the form

$$\frac{dy}{dx} = \pm\sqrt{-\frac{a_{22}}{a_{11}}}, \qquad \frac{a_{22}}{a_{11}} < 0 \tag{A.9}$$

and define a family of curves C_1 with a positive slope $dy / dx > 0$ and a family of curves C_2 with a negative slope $dy / dx < 0$. It can be shown that through each point, except the corner points, of the rectangular region Ω $(a \leqslant x \leqslant b,\ c \leqslant y \leqslant d)$ there pass two characteristics C_1 and C_2, while through the corner points, one characteristic C_1 or C_2.

Let us compute the derivatives dx / ds, dy / ds and the coefficients A_i on the characteristics. Directing s towards the increasing y, we have

$$\frac{dx}{ds} = \pm\left(1 - \frac{a_{22}}{a_{11}}\right)^{-1/2}, \qquad \frac{dy}{ds} = \left(1 - \frac{a_{11}}{a_{22}}\right)^{-1/2} \tag{A.10}$$
$$A_{2C_\alpha} = \pm\sqrt{-a_{11}a_{22}}, \qquad A_{3C_\alpha} = a_{11} + a_{22}$$

Here the upper sign is taken for the family C_1 $(\alpha = 1)$, the lower, for the family C_2 $(\alpha = 2)$. Carrying out analogous computations for the boundary lines, we find:

on the line $y = c$ (counting s in the direction of increasing x)

$$dx/ds = 1,\ dy / ds = 0,\ A_{1c} = a_{22}(x, c),\ A_{3c} = a_{11}(x, c) \tag{A.11}$$

on the line $y = d$ (counting s in the direction of decreasing x)

$$dx / ds = -1,\ dy / ds = 0,\ A_{1d} = a_{22}(x, d),\ A_{3d} = a_{11}(x, d) \tag{A.12}$$

on the line $x = a$ (counting s in the direction of decreasing y)

$$dx / ds = 0,\ dy / ds = -1,\ A_{1a} = a_{11}(a, y),\ A_{3a} = a_{22}(a, y) \tag{A.13}$$

on the line $x = b$ (counting s in the direction of increasing y)

$$dx / ds = 0,\ dy / ds = 1,\ A_{1b} = a_{11}(b, y),\ A_{3b} = a_{22}(b, y) \tag{A.14}$$

Let us consider the variation of the families of characteristics C_1 and C_2. Suppose that the equations

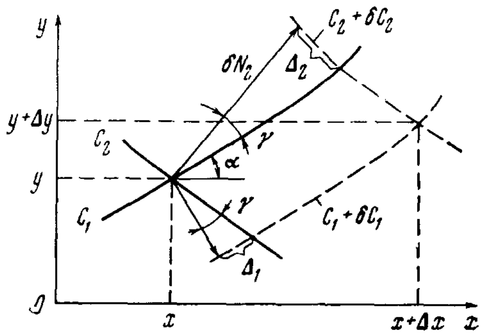

Fig. 5

$$F_\alpha(x, y) = D_\alpha \qquad (\alpha = 1, 2) \qquad (\text{A.}15)$$

for the families C_1 and C_2 have been found. Then the variation of these families depend upon the variation of the constants D_1 and D_2. Retaining only the terms of the first order of smallness, we obtain

$$(F_{\alpha x} n_{1C_\alpha} + F_{\alpha y} n_{2C_\alpha})\,\delta N_\alpha = \delta D_\alpha \qquad (\text{A.}16)$$

From this we obtain δN_1 and δN_2. If we differentiate δN_α with respect to s_α, and A_1 with respect to N_α, then for C_1 and C_2 we can establish the relation

$$2A_2\delta N_{\bullet} = A_{1N}\,\delta N \qquad (\text{A.}17)$$

Let us consider the variation of the point of intersection of the characteristics C_1 and C_2 (Fig. 5). To within terms of the first order of smallness we have

$$\begin{aligned}\Delta x\,|_{M_{ij}} &= n_{1C_1}\delta N_1 - n_{2C_1}\Delta_1 + n_{1C_2}\delta N_2 + n_{2C_2}\Delta_2 \\ \Delta y\,|_{M_{ij}} &= n_{2C_1}\delta N_1 + n_{1C_1}\Delta_1 + n_{2C_2}\delta N_2 - n_{1C_2}\Delta_2\end{aligned} \qquad (\text{A.}18)$$

From Fig. 5 we find

$$\begin{gathered}\operatorname{tg}\gamma = \frac{n_{1C_1}^2 - n_{2C_1}^2}{2n_{1C_1}n_{2C_1}} = -\frac{n_{1C_2}^2 - n_{2C_2}^2}{2n_{1C_2}n_{2C_2}} \\ \Delta_\alpha = \operatorname{tg}\gamma\,\delta N_\alpha \qquad (\alpha = 1, 2)\end{gathered} \qquad (\text{A.}19)$$

Substituting expressions (A.19) into (A.18), we obtain

$$\begin{aligned}\Delta x\,|_{M_{ij}} &= \frac{1}{2n_{1C_1}}\,\delta N_1 + \frac{1}{2n_{1C_2}}\,\delta N_2 \\ \Delta y\,|_{M_{ij}} &= \frac{1}{2n_{2C_1}}\,\delta N_1 + \frac{1}{2n_{2C_2}}\,\delta N_2\end{aligned} \qquad (\text{A.}20)$$

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